Higher-order Mathematical Operational Semantics aka Higher-order Abstract GSOS

Towards a unifying theory of operational methods/logical relations

Stelios Tsampas plus collaborators (mainly) from Erlangen

December 21, 2024

Friedrich-Alexander-Universität Erlangen-Nürnberg

Patterns in Formal (Operational) Methods

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2. Prove property X for a class of languages A

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Generalize

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Unify

MOS, aka Abstract GSOS, an exercise in composing mathematical concepts

$$\frac{p \xrightarrow{a} p'}{p \mid\mid q \xrightarrow{a} p' \mid\mid q}$$

Structural Op. Sem.

$$\begin{tikzcd}
\Sigma X \arrow[r, "g"] \arrow[d, "\Sigma(\mathrm{id}, h)"] & X \arrow[r, "h"] & BX \\
\Sigma(X \times BX) \arrow[rr, "\rho_X"] & & B(\Sigma^\star X) \arrow[u, "B(g^\star)"']
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Bialgebras

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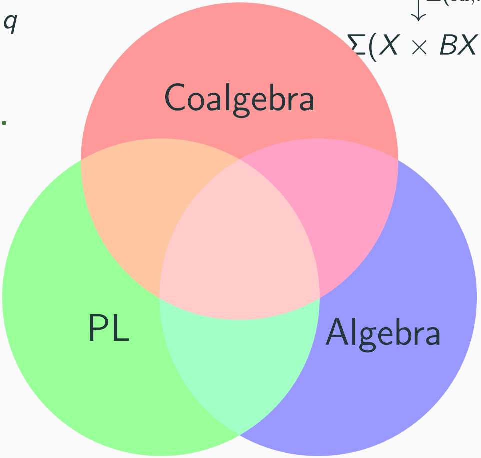

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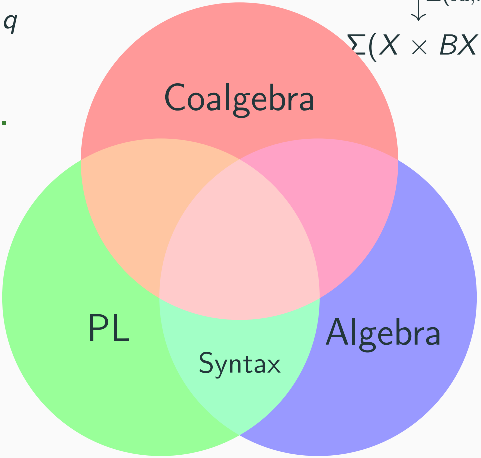

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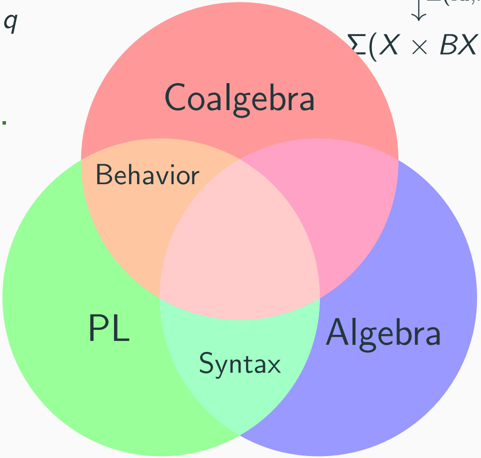

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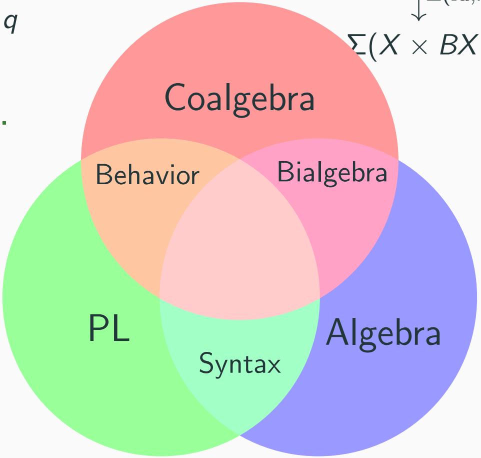

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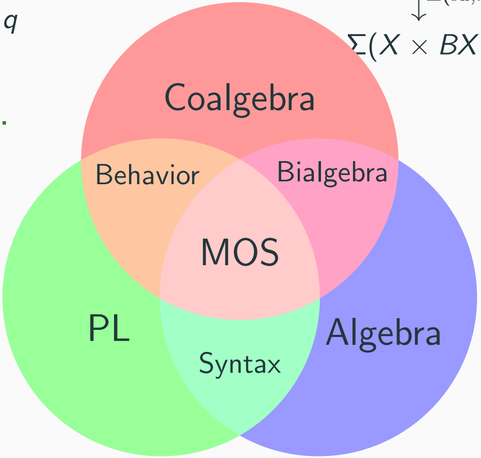

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- ☺ Great with (first-order) process calculi [1]–[4]
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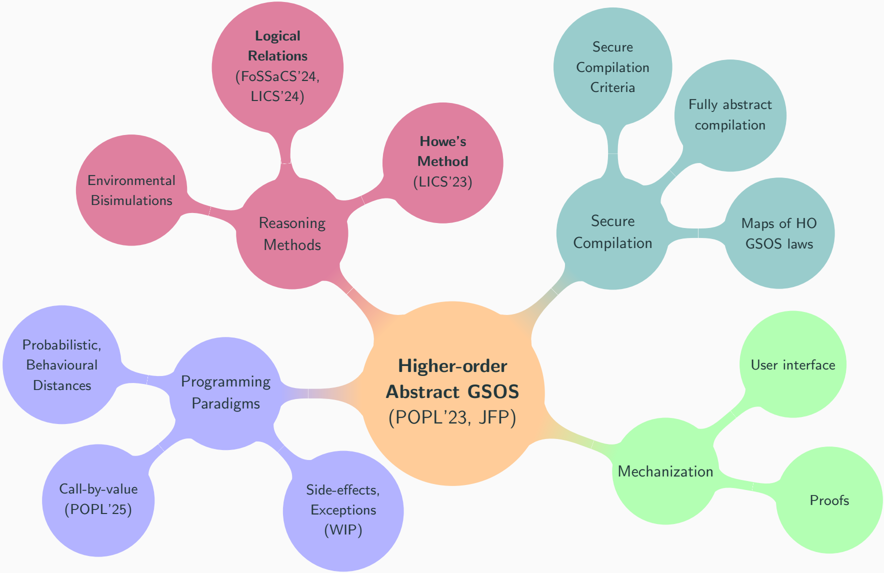
Higher-Order Abstract GSOS, a research programme



Overview

HO-MOS or
Higher-order Abstract GSOS

Relational Reasoning

Abstract Reasoning with
Step-indexed Logical Relations

HO-MOS or Higher-order Abstract GSOS

Abstract GSOS

Operational rules

$$\frac{p \xrightarrow{a} p'}{p \,||\, q \xrightarrow{a} p' \,||\, q}$$

$\cong$

GSOS laws: natural transformations

$$\rho_X \colon \underbrace{\Sigma(X \times BX)}_{\text{premises}} \to \underbrace{B(\Sigma^* X)}_{\text{conclusion}}$$

for functors $\Sigma, B \colon \mathcal{C} \to \mathcal{C}$ representing syntax and behaviour (e.g. $B = \mathcal{P}_{\mathsf{f}}^{\mathsf{L}}$).

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(coinductive) behaviours

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$$p_i \sim q_i \quad (i = 1, \ldots, n) \quad \xRightarrow{f \in \Sigma} \quad \mathsf{f}(p_1, \ldots, p_n) \sim \mathsf{f}(q_1, \ldots, q_n).$$

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- **Scope**: first-order (CCS, π-calculus, . . .), ~~higher-order~~ (λ-calculus, SKI calculus)

From first-order to higher-order

Higher-order languages require behaviours like $BX = X^X$.

This is not an endofunctor – but

$$B(X, Y) = Y^X$$

is a bifunctor contravariant in X and covariant in Y.

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Key idea for higher-order abstract GSOS [1]

$$\text{endofunctors } B\colon \mathcal{C} \to \mathcal{C} \quad + \quad \text{natural} \quad \text{transformations}$$

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[1] That part was straightfoward, the modelling of the λ-calculus and the compositionality of the semantics, not so much ☺.

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Figure 1: Operational semantics of the $\mathrm{SKI_u}$ calculus, our version of the SKI calculus, invented by Curry [10]. $\mathrm{SKI_u}$ in instance of an $\mathcal{HO}$ specification, a simple format of ours [5, §3].

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Disclaimer: This is just a convenient example to introduce HO-MOS. The latter can do the λ-calculus, typed or untyped, with simple or recursive types, etc.

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Labels can be a terms!

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Higher-order GSOS laws

Definition

A *higher-order GSOS law* of $\Sigma\colon \mathcal{C} \to \mathcal{C}$ (modelling the syntax) over $B\colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{C}$ (modelling higher-order behaviour) is a family of morphisms

$$\rho_{X,Y}\colon \Sigma(X \times B(X,Y)) \to B(X, \Sigma^{\star}(X+Y))$$

dinatural in $X \in \mathcal{C}$ and **natural** in $Y \in \mathcal{C}$.

Proposition

For every finitary signature $\check{\Sigma}$, with associated endofunctor $\Sigma\colon \mathsf{Set} \to \mathsf{Set}$, $\mathcal{HO}$ specifications are in a bijective correspondence with higher-order GSOS laws of Σ over $B(X, Y) = Y + Y^X$.

$$\frac{p \xrightarrow{q} p'}{p \quad q \longrightarrow p'}$$

$$\cong$$

$$\rho_X \colon \coprod_{\mathsf{f} \in \check{\Sigma}} (X \times (Y + Y^X))^{\mathsf{ar}(\mathsf{f})} \to \Sigma^*(X + Y)$$

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Higher-Order Abstract GSOS

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Higher-order GSOS laws: (di-)natural trf.

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For combinator calculi, we have

$$\mathcal{C} = \mathsf{Set}$$

$$\Sigma X = 1 + X \times X + \dots$$

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β-reduction or combinator

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$$\frac{}{(\lambda x.p)\,q \to p[q/x]}$$

$\cong^\star$

Higher-order GSOS laws: (di-)natural trf.

$$\rho_{X,Y} \colon \underbrace{\Sigma(X \times B(X,Y))}_{\text{premises}} \to \underbrace{B(X, \Sigma^*(X+Y))}_{\text{conclusion}}$$

For the call-by-name λ-calculus, we have

$$\mathcal{C} = \mathrm{Set}^{\mathbb{F}}$$

$$\Sigma X = V + \delta X + X \times X \quad \text{(Fiore, Plotkin and Turi [11])}$$

$$B(X,Y) = \langle X, Y \rangle \times (Y + Y^X + 1)$$

substitution stucture

β-reduction, λ-expr or stuck

Higher-Order Abstract GSOS

Operational rules

$$\frac{}{(\lambda x.p)\,q \to p[q/x]}$$

$$\frac{p \to p'}{p\,q \to p'\,q}$$

$\cong^\star$

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Higher-Order Abstract GSOS

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---

programs

- Operational model $\gamma : \mu\Sigma \to B(\mu\Sigma, \mu\Sigma)$, ~~denotational model~~.

  e.g. $\gamma(t) = t'$ if $t \to t'$ and $\gamma(\lambda x.M) = (e \mapsto M[e/x])$, $(\gamma(I) = \mathrm{id}$ for SKI)

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Operational rules

$$\frac{}{(\lambda x.p)\, q \to p[q/x]}$$

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  e.g. $\gamma(t) = t'$ if $t \to t'$ and $\gamma(\lambda x.M) = (e \mapsto M[e/x])$, $(\gamma(I) = \mathrm{id}$ for SKI)

- $\underbrace{\textbf{Key feature: } \text{compositionality, i.e. bisimilarity is a congruence.}}_{\text{Proof: more complex than first-order case + needs mild assumptions.}}$

Strong Applicative Bisimilarity

Coalgebraic bisimilarity on operational model $\mu\Sigma \to B(\mu\Sigma, \mu\Sigma)$

$$=$$

strong applicative bisimilarity.

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Example: λ-calculus

closed λ-terms

Greatest relation $\sim \subseteq \Lambda \times \Lambda$ such that for $t_1 \sim t_2$,

$$t_1 \to t_1' \implies t_2 \to t_2' \quad \wedge \; t_1' \sim t_2';$$
$$t_1 = \lambda x.t_1' \implies t_2 = \lambda x.t_2' \; \wedge \; \forall e \in \Lambda.\; t_1'[e/x] \sim t_2'[e/x];$$

$+$ two symmetric conditions

Abstract modelling of Operational Semantics

Concrete/Abstract

Abstract modelling of Operational Semantics

Concrete/Abstract

1. Algebraic signature $\check{\Sigma}$

Abstract modelling of Operational Semantics

Concrete/Abstract

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1. Syntax endofunctor $\Sigma\colon \mathcal{C} \to \mathcal{C}$

Abstract modelling of Operational Semantics

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2. Program terms $\mu\Sigma$

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2. Initial Σ-algebra $\mu\Sigma = \Sigma^*(0)$

Abstract modelling of Operational Semantics

Concrete/Abstract

1. Algebraic signature $\check{\Sigma}$
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3. (Impl.) nature of computation

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Concrete/Abstract

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Abstract modelling of Operational Semantics

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4. Operational rules $\frac{t \to t'}{t \cdot s \to t' \cdot s}$

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4. Operational rules $\dfrac{t \to t'}{t \cdot s \to t' \cdot s}$	4. Higher-order GSOS law $\rho_{X,Y}$

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5. Oper. model $t \to t'$, $t, t' \in \mu\Sigma$

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Assuming a suitable category $\mathcal{C}$.

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Assuming a suitable category $\mathcal{C}$.

[5]: Congruence of bisimilarity, for free!

Question marks

Concrete/Abstract

Question marks

Concrete/Abstract

8. Howe's closure
9. Howe's method
10. Logical predicates/relations
11. Fundamental Properties

Question marks

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Concrete/Abstract	
8. Howe's closure	8. ???
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We want to model all of the above **generically**, in a **language-independent** manner.

Question marks

Concrete/Abstract

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We want to mode… …ove **generically**, in… …**ndependent** manner.

Relation Lifting!

Predicate Lifting!

Relational Reasoning

How to do program discourse, categorically

Key concept 1: If $\mathcal{C}$ is our base universe of discourse, we can form the categories $\mathsf{Rel}(\mathcal{C})$ and $\mathsf{Pred}(\mathcal{C})$ of resp. (homogenous) relations and predicates on $\mathcal{C}$. These are the categories of subobjects on rep. $X \times X$ and X.

$$\begin{tikzcd} R \arrow[r, dashed] \arrow[d, tail, "\langle \mathsf{l}_R, \mathsf{r}_R \rangle"'] & S \arrow[d, tail, "\langle \mathsf{l}_S, \mathsf{r}_S \rangle"] \\ X \times X \arrow[r, "f \times f"] & Y \times Y \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, dashed] \arrow[d, tail, "p"'] & Q \arrow[d, tail, "q"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

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Key concept 2: We extend functors (and the rest of the constructions) to $\mathsf{Rel}(\mathcal{C})$ and $\mathsf{Pred}(\mathcal{C})$, a process that is known as relation (or predicate) lifting [12].

$$\begin{tikzcd} \mathsf{Rel}(\mathcal{C}) \arrow[r, "\overline{\Sigma}"] \arrow[d, "|-|"'] & \mathsf{Rel}(\mathcal{C}) \arrow[d, "|-|"] \\ \mathcal{C} \arrow[r, "\Sigma"] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Rel}(\mathcal{C})^{\mathsf{op}} \times \mathsf{Rel}(\mathcal{C}) \arrow[r, "\overline{B}"] \arrow[d, "|-|^{\mathsf{op}} \times |-|"'] & \mathsf{Rel}(\mathcal{C}) \arrow[d, "|-|"] \\ \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \arrow[r, "B"] & \mathcal{C} \end{tikzcd}$$

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Key concept 1: If $\mathcal{C}$ is our base universe of discourse, we can form the categories $\mathsf{Rel}(\mathcal{C})$ and $\mathsf{Pred}(\mathcal{C})$ of resp. (homogenous) relations and predicates on $\mathcal{C}$. These are the categories of subobjects on rep. $X \times X$ and X.

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Also, write $\mathsf{Pred}_X(\mathcal{C})$, $\mathsf{Rel}_X(\mathcal{C})$ for the lattices of resp. predicates and relations on X.

Act I, Induction. Part 1, Predicates.

Let $P \rightarrowtail \mu\Sigma$ be a predicate on terms (assume a typed syntax, for the heck of it).

Structural induction

1. (Repeat for every operation) For all $t : \tau_1 \rightarrow \tau_2$, $s : \tau_1$ such that $P_{\tau_1 \rightarrow \tau_2}(t)$ and $P_{\tau_1}(s)$, then $P_{\tau_2}(t\, s)$.
2. By induction, for all types τ and terms $t : \tau$, $P_\tau(t)$.

Unary induction proof principle

1. $\overline{\Sigma}(P)$ represents 1-depth terms (operations) whose subterms are in P ($\overline{\Sigma}$ is the <u>canonical</u> lifting). There is a Σ-algebra structure

$$\overline{\Sigma}(P) \leq \iota^\star[P], \text{ where } \iota \colon \Sigma\mu\Sigma \to \mu\Sigma \text{ is the initial } \Sigma\text{-algebra.}$$

2. As initial algebras have no proper subalgebras, $P \cong \mu\Sigma$.

Act I, Induction. Part 2, Relations.

Let $R \rightarrowtail \mu\Sigma \times \mu\Sigma$ be a relation on terms.

Structural induction (Fundamental Property)

1. (Repeat for every operation) For all $t_1, t_2 : \tau_1 \rightarrow \tau_2$, $s_1, s_2 : \tau_1$ such that $R_{\tau_1 \rightarrow \tau_2}(t_1, t_2)$ and $R_{\tau_1}(s_1, s_2)$, then $R_{\tau_2}(t_2\, s_2, t_2\, s_2)$.
2. Then for all types τ, relation R_τ is reflexive.

Binary induction proof principle

1. $\overline{\Sigma}(R)$ represents pairs of 1-depth terms with subterms in R. If there is

$$\overline{\Sigma}(R) \leq (\iota \times \iota)^\star[R](\text{that is, } R \text{ is a congruence}),$$

2. then $\Delta \leq R$ because all congruences on an initial algebra are reflexive.

Simple go-to example (untyped syntax this time)

$$B(X,Y) : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{C} \quad \gamma\colon \mu\Sigma \to B(\mu\Sigma,\mu\Sigma)$$

$$B(X,Y) = Y + Y^X \qquad \gamma(t) = t' \text{ if } t \to t' \text{ and } \gamma(\lambda x.M) = (e \mapsto M[e/x])$$

Act II, Bisimulations. Prelude.

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$$\begin{tikzcd} \mathsf{Pred}(\mathcal{C})^{\mathrm{op}} \times \mathsf{Pred}(\mathcal{C}) \arrow[r, "\overline{B}"] \arrow[d, "|-|^{\mathrm{op}} \times |-|"'] & \mathsf{Pred}(\mathcal{C}) \arrow[d, "|-|"] \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[r, "B"] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Rel}(\mathcal{C})^{\mathrm{op}} \times \mathsf{Rel}(\mathcal{C}) \arrow[r, "\overline{B}"] \arrow[d, "|-|^{\mathrm{op}} \times |-|"'] & \mathsf{Rel}(\mathcal{C}) \arrow[d, "|-|"] \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[r, "B"] & \mathcal{C} \end{tikzcd}$$

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\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[r, "B"] & \mathcal{C}
\end{tikzcd}$$

Let $R, S \subseteq \mu\Sigma \times \mu\Sigma$ be relations. Then $\overline{B}(R,S)$ amounts to the following:

$$\overline{B}(R,S) = \{(t_1,t_2) \mid Q(t_1,t_2)\} \vee \{f \in \mu\Sigma^{\mu\Sigma} \mid \forall t_1, t_2.\, R(t_1,t_2) \implies Q(f(t_1), f(t_2))\},$$

aka, related inputs are mapped to related outputs!

Act II, Bisimulations. Logical Relations.

Let R be a relation on the state space of a coalgebra $h : X \to B(X, X)$. We say that R is a logical relation (for h, h) if

$$R \leq h^{\star}[\overline{B}(R, R)]$$

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Instantiate on $\gamma : \mu\Sigma \to B(\mu\Sigma, \mu\Sigma)$. A relation R is logical if the following hold for all t, s with $R(t, s)$:

1. If $t = \lambda x.t'$, then $s = \lambda x.s'$ and

   for all terms e_1, e_2 with $R(e_1, e_2)$, we have $R(t'[e_1/x], s'[e_2/x])$.

2. If $t \to t'$ then $s \to s'$ and $R(t', s')$.

Act II, Bisimulations. Logical Relations.

Let R be a relation on the state space of a coalgebra $h : X \to B(X, X)$. We say that R is a logical relation (for h, h) if

$$R \leq h^\star[\overline{B}(R, R)]$$

If R is logical, then the following is true for all $t, s \colon \sigma \rightarrowtriangle \tau$ with $R_{\sigma \rightarrowtriangle \tau}(t, s)$:

1. If $t = \lambda x \colon \sigma.t'$, then $s = \lambda x \colon \sigma.s'$ and

   $$\text{for all terms } e_1, e_2 \colon \sigma \text{ with } R_\sigma(e_1, e_2), \text{ we have } R_\tau(t'[e_1/x], s'[e_2/x]).$$

2. If $t \to t'$ then $s \to s'$ and $R_{\sigma \rightarrowtriangle \tau}(t', s')$.

Bisimulations, logical relations and step-indexing [8]

Let $h : X \to B(X, X)$ be a coalgebra and $\tilde{h} : X \to B(X, X)$ be a weakening of h (think $\to$ vs its saturation/closure $\Rightarrow$). We say that:

1. A relation R on X is a **($\overline{B}$-)bisimulation** (for $h, \tilde{h}$) if $R \leq (h \times \tilde{h})^{\star}[\overline{B}(\Delta, R)]$.
2. A relation R on X is a **($\overline{B}$-)logical relation** (for $h, \tilde{h}$) if $R \leq (h \times \tilde{h})^{\star}[\overline{B}(R, R)]$.
3. An ordinal-indexed family of relations $(R^{\alpha} \rightarrowtail X \times X)_{\alpha}$ is a **($\overline{B}$-)step-indexed logical relation** (for $h, \tilde{h}$) if it forms a decreasing chain (i.e. $R^{\alpha} \leq R^{\beta}$ for all $\beta < \alpha$) and satisfies

$$R^{\alpha+1} \leq (h \times \tilde{h})^{\star}[\overline{B}(R^{\alpha}, R^{\alpha})] \quad \text{for all } \alpha.$$

Abstract modelling of Predicates and Relations

Concrete/Abstract

Abstract modelling of Predicates and Relations

Concrete/Abstract

1. Predicates, relations on terms

Abstract modelling of Predicates and Relations

Concrete/Abstract

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1. $P \rightarrowtail \mu\Sigma$, $R \rightarrowtail \mu\Sigma \times \mu\Sigma$

Abstract modelling of Predicates and Relations

Concrete/Abstract

1. Predicates, relations on terms
2. Predicate, relational reasoning

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Abstract modelling of Predicates and Relations

Concrete/Abstract

1. Predicates, relations on terms
2. Predicate, relational reasoning

1. $P \rightarrowtail \mu\Sigma$, $R \rightarrowtail \mu\Sigma \times \mu\Sigma$
2. Complete, well-powered cat. $\mathcal{C}$

Abstract modelling of Predicates and Relations

Concrete/Abstract

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1. Predicates, relations on terms	1. $P \rightarrowtail \mu\Sigma$, $R \rightarrowtail \mu\Sigma \times \mu\Sigma$
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3. (P is a) Logical Predicate	

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3. (P is a) Logical Predicate	3. $P \leq h^{\star}[\overline{B}(P, P)]$

Abstract modelling of Predicates and Relations

Concrete/Abstract

1. Predicates, relations on terms
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3. (P is a) Logical Predicate
4. (R is a) Logical Relation

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1. Predicates, relations on terms	1. $P \rightarrowtail \mu\Sigma$, $R \rightarrowtail \mu\Sigma \times \mu\Sigma$
2. Predicate, relational reasoning	2. Complete, well-powered cat. $\mathcal{C}$
3. (P is a) Logical Predicate	3. $P \leq h^\star[\overline{B}(P, P)]$
4. (R is a) Logical Relation	4. $R \leq (h \times \tilde{h})^\star[\overline{B}(R, R)]$

Abstract modelling of Predicates and Relations

Concrete/Abstract

1. Predicates, relations on terms
2. Predicate, relational reasoning
3. (P is a) Logical Predicate
4. (R is a) Logical Relation
5. Fundamental Property
   of Logical Relations

1. $P \rightarrowtail \mu\Sigma$, $R \rightarrowtail \mu\Sigma \times \mu\Sigma$
2. Complete, well-powered cat. $\mathcal{C}$
3. $P \leq h^\star[\overline{B}(P,P)]$
4. $R \leq (h \times \tilde{h})^\star[\overline{B}(R,R)]$

Abstract modelling of Predicates and Relations

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Recall that relation lifting is algebraic and coalgebraic, and independent of the Higher-order Abstract GSOS framework.

However, the marriage of algebra and coalgebra that HO Abstract GSOS represents extends along their liftings :).

Abstract Reasoning with Step-indexed Logical Relations

Constructing step-indexed logical relation, Coalgebraically

Let's systematize the construction of a step-indexed logical relation, in a language-independent manner.

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Step-indexed Henceforth Relation Transformer

Let $B\colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{C}$ with a relation lifting $\overline{B}$, and let $c, \tilde{c}\colon X \to B(X,X)$ be coalgebras. For every $R \rightarrowtail X \times X$ we define the step-indexed logical relation $(\,\Box^{\overline{B},c,\tilde{c},\alpha} R \rightarrowtail X \times X\,)_\alpha$ by transfinite induction (writing $\Box^\alpha$ for simplicity):

$$
\begin{aligned}
\Box^0 R &= R,\\
\Box^{\alpha+1} R &= \Box^\alpha R \wedge (c \times \tilde{c})^\star[\overline{B}(\Box^\alpha R, \Box^\alpha R)],\\
\Box^\alpha R &= \bigwedge_{\beta<\alpha} \Box^\beta R \qquad \text{for limit ordinals } \alpha.
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Let's systematize the construction of a step-indexed logical relation, in a language-independent manner.

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\square^\alpha R &= \bigwedge_{\beta<\alpha} \square^\beta R \qquad \text{for limit ordinals } \alpha.
\end{aligned}$$

Under mild conditions, there exists ν with $\square^{\nu+1} R = \square^\nu R$, which makes $\square^\nu R$ logical. For **the** logical relation, the "chosen one", plug $R = \top = X \times X$.

Constructing step-indexed logical relation, Coalgebraically

$$\mathcal{L}^0_\tau(\Gamma) = \top_\tau(\Gamma) = \{(t,s) \mid \Gamma \vdash t, s\colon \tau\}$$

$$\mathcal{L}^{\alpha+1}_\tau = \mathcal{L}^\alpha_\tau \cap \mathcal{S}_\tau(\mathcal{L}^\alpha, \mathcal{L}^\alpha) \cap \mathcal{E}_\tau(\mathcal{L}^\alpha) \cap \mathcal{V}_\tau(\mathcal{L}^\alpha, \mathcal{L}^\alpha)$$

$$\mathcal{L}^\alpha_\tau(\Gamma) = \bigcap_{\beta<\alpha} \mathcal{L}^\alpha_\tau(\Gamma) \quad \text{for limit ordinals } \alpha.$$

$$\mathcal{S}_\tau(\Gamma)(Q,R) = \{(t,s) \mid \text{for all } \Delta \text{ and } Q_{\Gamma(x)}(\Delta)(u_x, v_x)\ (x \in |\Gamma|), \text{ one has } R_\tau(\Delta)(t[\vec{u}], s[\vec{v}])\},$$

$$\mathcal{E}_\tau(\Gamma)(R) = \{(t,s) \mid \text{if } t \to t' \text{ then } \exists s'.\, s \Rightarrow s' \wedge R_\tau(\Gamma)(t',s')\},$$

$$\mathcal{V}_{\tau_1 \boxtimes \tau_2}(\Gamma)(Q,R) = \{(t,s) \mid \text{if } t = \mathsf{pair}_{\tau_1,\tau_2}(t_1,t_2) \text{ then } \exists s_1, s_2.\, s \Rightarrow \mathsf{pair}_{\tau_1,\tau_2}(s_1,s_2) \wedge R_{\tau_1}(\Gamma)(t_1,s_1) \wedge R_{\tau_2}(\Gamma)(t_2,s_2)\},$$

$$\mathcal{V}_{\mu\alpha.\tau}(\Gamma)(Q,R) = \{(t,s) \mid \text{if } t = \mathsf{fold}_\tau(t') \text{ then } \exists s'.\, s \Rightarrow \mathsf{fold}_\tau(s') \wedge R_{\tau[\mu\alpha.\tau/\alpha]}(\Gamma)(t',s')\},$$

$$\mathcal{V}_{\tau_1 \multimap \tau_2}(\Gamma)(Q,R) = \{(t,s) \mid \text{for all } Q_{\tau_1}(\Gamma)(e,e'), \text{ if } t = \lambda x.t' \text{ then } \exists s'.\, s \Rightarrow \lambda x.s' \ \wedge\ R_{\tau_2}(\Gamma)(t'[e/x], s'[e'/x])\}.$$

Logical Relations, abstractly

Data: Higher-Order GSOS law of Σ over B in a suitable category $\mathcal{C}$, liftings, weakening of the operational model (the coalgebra on terms $\mu\Sigma$) and mild conditions on $\mathcal{C}$.

Main theorem (informal)

Let $R \rightarrowtail \mu\Sigma \times \mu\Sigma$ be a congruence. Assuming a lax-bialgebra condition. If R is a congruence, then for all α, $\Box^\alpha R$ is a congruence.

$$\frac{t \overset{s}{\Rightarrow} t'}{t\,s \Rightarrow t'} \quad \checkmark \qquad\qquad \frac{t \Rightarrow t'}{t\,s \Rightarrow t'\,s} \quad \checkmark$$

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Corollary

1. *For all α, $\Box^\alpha \top$ is a congruence.*
2. *$\Box^\nu \top$ is a congruence (and hence reflexive) and, for "reasonable" definitions of contextual equivalence, sound w.r.t. contextual equivalence.*

The point of all this

Accept for a single slide that every higher-order operational semantics is a higher-order GSOS law. You are presented with such semantics and looking for a sensible logical relation, sound for contextual equivalence. What we're saying is that the actual work that needs to be done, the language-dependent part of the problem, is the following:

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The intuition is that the standard compatibility lemmas contain lots of boilerplate, contrived proof code that should be "automatic" under reasonable circumstances.

Distilling the essence of Operational Methods

By systematizing Howe's method and (step-indexed) logical relations, we show that, assuming the operational semantics are sane in certain way [2], then

1. Howe's method can be applied.
2. The evident logical relation should be sound w.r.t. contextual equivalence.

[2] They form a HO-GSOS law and a lax bialgebra.

Thank you!

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